

LEFSCHETZ FIXED POINT FORMULA ON A COMPACT RIEMANNIAN MANIFOLD WITH BOUNDARY FOR SOME BOUNDARY CONDITIONS

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ABSTRACT. In [8] the authors introduced a pair of new de Rham complexes on a compact oriented Riemannian manifold with boundary by using a pair of new boundary conditions to discuss the refined analytic torsion on a compact manifold with boundary. In this paper we discuss the Lefschetz fixed point formula on these complexes with respect to a smooth map having simple fixed points and satisfying some special condition near the boundary. For this purpose we are going to use the heat kernel method for the Lefschetz fixed point formula.

1. INTRODUCTION

Let (M, Y, g^M) be an m -dimensional compact oriented Riemannian manifold with boundary Y and $f : M \rightarrow M$ be a smooth map such that $f(Y) \subset Y$. A point $p \in M$ is said to be a simple fixed point of f if

$$f(p) = p, \quad \det(I - df(p)) \neq 0. \quad (1.1)$$

If p is a simple fixed point, the graph of f is transverse to the diagonal of $M \times M$ at (p, p) , which implies that simple fixed points are discrete. All through this paper we assume that all fixed points of f are simple and hence f has only finitely many fixed points. For fixed points on the boundary Y , we need one more structure. Let $f(x_0) = x_0$ with $x_0 \in Y$. Then $df(x_0) : T_{x_0}M \rightarrow T_{x_0}M$ induces a map $df_Y(x_0) : T_{x_0}Y \rightarrow T_{x_0}Y$. We consider

$$a_{x_0} = df(x_0)(\text{mod } T_{x_0}Y) : T_{x_0}M/T_{x_0}Y \rightarrow T_{x_0}M/T_{x_0}Y.$$

Since the quotient space $T_{x_0}M/T_{x_0}Y$ is one-dimensional, the map a_{x_0} is simply multiplication by a number, which we denote by a_{x_0} again. It's not difficult to see that $a_{x_0} \geq 0$ by considering the quotient space $T_{x_0}M/T_{x_0}Y$ as a normal half-line pointing inward at the boundary point x_0 . Moreover, since the fixed point x_0 is simple, $a_{x_0} \neq 1$ (see [5] for details).

Definition 1.1. (1) A simple boundary fixed point $x_0 \in Y$ is called *attracting* if $a_{x_0} < 1$ and *repelling* if $a_{x_0} > 1$.

(2) We denote by $\mathcal{F}_0(f)$, $\mathcal{F}_Y^+(f)$ and $\mathcal{F}_Y^-(f)$ the set of all simple fixed points in the interior of M ,

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the attracting fixed points in Y and the repelling fixed points in Y , respectively. We denote $\mathcal{F}_Y(f) := \mathcal{F}_Y^+(f) \cup \mathcal{F}_Y^-(f)$ and $\mathcal{F}(f) := \mathcal{F}_0(f) \cup \mathcal{F}_Y(f)$.

A. V. Brenner and M. A. Shubin proved the following result in [5].

$$\begin{aligned} \sum_{q=0}^m (-1)^q \operatorname{Tr} (f^* : H^q(M) \rightarrow H^q(M)) &= \sum_{p \in \mathcal{F}_0(f) \cup \mathcal{F}_Y^+(f)} \operatorname{sign} \det (I - df(p)), \\ \sum_{q=0}^m (-1)^q \operatorname{Tr} (f^* : H^q(M, Y) \rightarrow H^q(M, Y)) &= \sum_{p \in \mathcal{F}_0(f) \cup \mathcal{F}_Y^-(f)} \operatorname{sign} \det (I - df(p)). \end{aligned} \quad (1.2)$$

This result extends the Atiyah-Bott-Lefschetz fixed point formula proven on a closed manifold in [1].

On the other hand, the authors introduced new de Rham complexes $(\Omega_{\tilde{\mathcal{P}}_0}^\bullet(M), d)$ and $(\Omega_{\tilde{\mathcal{P}}_1}^\bullet(M), d)$ by using some boundary conditions $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$, which compute $H^q(\Omega_{\tilde{\mathcal{P}}_0}^\bullet(M), d) = \begin{cases} H^q(M, Y) & \text{if } q = \text{even} \\ H^q(M) & \text{if } q = \text{odd} \end{cases}$ and $H^q(\Omega_{\tilde{\mathcal{P}}_1}^\bullet(M), d) = \begin{cases} H^q(M) & \text{if } q = \text{even} \\ H^q(M, Y) & \text{if } q = \text{odd} \end{cases}$. In this paper, we are going to discuss the Lefschetz fixed point formula on these complexes. More precisely, when $f : M \rightarrow M$ is a smooth map having simple fixed points and satisfying some special condition near the boundary Y (see Definition 3.1), we are going to describe

$$\begin{aligned} \sum_{q=\text{even}} \operatorname{Tr} (f^* : H^q(M, Y) \rightarrow H^q(M, Y)) - \sum_{q=\text{odd}} \operatorname{Tr} (f^* : H^q(M) \rightarrow H^q(M)) &\quad \text{and} \\ \sum_{q=\text{even}} \operatorname{Tr} (f^* : H^q(M) \rightarrow H^q(M)) - \sum_{q=\text{odd}} \operatorname{Tr} (f^* : H^q(M, Y) \rightarrow H^q(M, Y)) \end{aligned}$$

in terms of fixed points of f and some additional data (see Theorem 3.4 below). For this purpose, we are going to use the heat kernel method for the Lefschetz fixed point formula (cf. [3], [6]).

2. DE RHAM COMPLEX $(\Omega_{\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1}^\bullet(M), d)$ ON A COMPACT RIEMANNIAN MANIFOLD WITH BOUNDARY

In this section we are going to introduce the de Rham complex $(\Omega_{\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1}^\bullet(M), d)$ on a compact Riemannian manifold with boundary by using the boundary condition $\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1$. We recall that (M, Y, g^M) is an m -dimensional compact oriented Riemannian manifold with boundary Y . From now on, we assume that g^M is a product metric near the boundary Y . We denote by $d_q^Y : \Omega^q(Y) \rightarrow \Omega^{q+1}(Y)$ the de Rham operator induced from $d : \Omega^q(M) \rightarrow \Omega^{q+1}(M)$ and denote by $\star_Y : \Omega^q(Y) \rightarrow \Omega^{m-1-q}(Y)$ the Hodge star operator on Y induced from the Hodge star operator \star_M on M . Then the formal adjoint $(d_q^Y)^*$ of d_q^Y is defined in the usual way. We denote $\Delta_Y^q := (d_q^Y)^* d_q^Y + d_{q-1}^Y (d_{q-1}^Y)^*$ and $\mathcal{H}^q(Y) := \ker \Delta_Y^q$. By the Hodge decomposition, we have

$$\Omega^q(Y) = \operatorname{Im} d_{q-1}^Y \oplus \mathcal{H}^q(Y) \oplus \operatorname{Im} (d_q^Y)^*$$

Let N be a collar neighborhood of Y which is isometric to $[0, 1) \times Y$ and u be the coordinate normal to the boundary Y on N . If $d\phi = d^*\phi = 0$ for $\phi \in \Omega^q(M)$, simple computation shows that ϕ is expressed on the boundary Y by

$$\phi|_Y = (d^Y \varphi_1 + \varphi_2) + du \wedge (d^{Y*} \psi_1 + \psi_2), \quad \varphi_1, \psi_1 \in \Omega^\bullet(Y), \quad \varphi_2, \psi_2 \in \mathcal{H}^\bullet(Y). \quad (2.1)$$

In other words, φ_2 and ψ_2 are harmonic parts of $\iota^*\phi$ and $\star_Y \iota^*(\star_M \phi)$ up to sign, where $\iota : Y \rightarrow M$ is the natural inclusion. We denote \mathcal{K}^q and \mathcal{K} by

$$\mathcal{K}^q := \{\varphi_2 \in \mathcal{H}^q(Y) \mid d\phi = d^*\phi = 0\}, \quad \mathcal{K} := \bigoplus_{q=0}^{m-1} \mathcal{K}^q, \quad (2.2)$$

where ϕ has the form (2.1). If $d\phi = d^*\phi = 0$ for $\phi \in \Omega^q(M)$, $d(\star_M \phi) = d^*(\star_M \phi) = 0$, which implies that

$$\star_Y \mathcal{K}^{m-q} = \{\psi_2 \in \mathcal{H}^{q-1}(Y) \mid d\phi = d^*\phi = 0\}, \quad (2.3)$$

where ϕ has the form (2.1). We have the following lemma, whose proof we refer to Lemma 2.4 in [8].

Lemma 2.1. *\mathcal{K} is orthogonal to $\star_Y \mathcal{K}$ and $\mathcal{K} \oplus (\star_Y \mathcal{K}) = \mathcal{H}^\bullet(Y)$.*

We consider the homomorphism $\iota^* : H^\bullet(M) \rightarrow H^\bullet(Y)$ induced from the natural inclusion $\iota : Y \rightarrow M$. It is well known that each cohomology class $[\omega] \in H^\bullet(M)$ has a unique representative $\omega_0 \in \Omega^\bullet(M)$ such that $d\omega_0 = d^*\omega_0 = 0$ and $\iota^*(\star_M \omega_0) = 0$ (see Theorem 2.7.3 in [6]). Since $\iota^*\omega_0$ is a closed form, $[\iota^*\omega_0] \in H^\bullet(Y)$. We denote by $(\iota^*\omega_0)_h$ the harmonic part of $\iota^*\omega_0$ and define a map

$$\mathcal{G} : \text{Im}(\iota^* : H^\bullet(M) \rightarrow H^\bullet(Y)) \rightarrow \mathcal{K}, \quad \mathcal{G}([\iota^*\omega_0]) = (\iota^*\omega_0)_h.$$

A standard argument using the Lefschetz-Poincaré duality shows that $\dim \text{Im}(\iota^* : H^\bullet(M) \rightarrow H^\bullet(Y))$ is equal to $\frac{1}{2} \dim H^\bullet(Y)$. Since \mathcal{G} is a monomorphism, this fact together with Lemma 2.1 shows that \mathcal{G} is an isomorphism. Summarizing this fact, we have the following result (cf. Corollary 8.4 in [9]).

Lemma 2.2. *For each q , \mathcal{K}^q can be naturally identified with $\text{Im}(\iota^* : H^q(M) \rightarrow H^q(Y))$.*

We next consider the natural isomorphism

$$\Psi : \Omega^p(N) \rightarrow C^\infty([0, 1), \Omega^p(Y) \oplus \Omega^{p-1}(Y)), \quad \Psi(\omega_1 + du \wedge \omega_2) = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \quad (2.4)$$

We put $\mathcal{L}_0 := \begin{pmatrix} \mathcal{K} \\ \mathcal{K} \end{pmatrix}$, $\mathcal{L}_1 := \begin{pmatrix} \star_Y \mathcal{K} \\ \star_Y \mathcal{K} \end{pmatrix}$ and consider the orthogonal projections defined by

$$\mathcal{P}_{-, \mathcal{L}_0}, \mathcal{P}_{+, \mathcal{L}_1} : \Omega^\bullet(Y) \oplus \Omega^\bullet(Y) \rightarrow \Omega^\bullet(Y) \oplus \Omega^\bullet(Y) \\ \text{Im } \mathcal{P}_{-, \mathcal{L}_0} = \begin{pmatrix} \text{Im } d^Y \oplus \mathcal{K} \\ \text{Im } d^Y \oplus \mathcal{K} \end{pmatrix}, \quad \text{Im } \mathcal{P}_{+, \mathcal{L}_1} = \begin{pmatrix} \text{Im}(d^Y)^* \oplus \star_Y \mathcal{K} \\ \text{Im}(d^Y)^* \oplus \star_Y \mathcal{K} \end{pmatrix}.$$

We then define the spaces of differential forms satisfying the boundary conditions $\mathcal{P}_{-, \mathcal{L}_0}$ and $\mathcal{P}_{+, \mathcal{L}_1}$ by

$$\begin{aligned} \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^q(M) &:= \{\phi \in \Omega^q(M) \mid \mathcal{P}_{-, \mathcal{L}_0}(\phi|_Y) = 0, \quad \mathcal{P}_{-, \mathcal{L}_0}((\star_M(d + d^*)\phi)|_Y) = 0\}, \\ \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^q(M) &:= \{\phi \in \Omega^q(M) \mid \mathcal{P}_{+, \mathcal{L}_1}(\phi|_Y) = 0, \quad \mathcal{P}_{+, \mathcal{L}_1}((\star_M(d + d^*)\phi)|_Y) = 0\}, \end{aligned}$$

and also define

$$\begin{aligned}\Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{q, \infty}(M) &= \{\phi \in \Omega^q(M) \mid \mathcal{P}_{-, \mathcal{L}_0}((\star_M(d + d^*))^l \phi)|_Y) = 0, \quad l = 0, 1, 2, \dots\}, \\ \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{q, \infty}(M) &= \{\phi \in \Omega^q(M) \mid \mathcal{P}_{+, \mathcal{L}_1}((\star_M(d + d^*))^l \phi)|_Y) = 0, \quad l = 0, 1, 2, \dots\}.\end{aligned}\quad (2.5)$$

Simple computation shows that if $\phi \in \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^q(M)$, then $\star_M \phi \in \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{m-q}(M)$ and vice versa. Similarly, for each $\phi \in \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^q(M)$ and $\psi \in \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^q(M)$, we have

$$\mathcal{P}_{+, \mathcal{L}_1}((d\phi)|_Y) = 0 \quad \text{and} \quad \mathcal{P}_{-, \mathcal{L}_0}((d\psi)|_Y) = 0. \quad (2.6)$$

These imply that \star_M maps $\Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{q, \infty}(M)$ ($\Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{q, \infty}(M)$) into $\Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{m-q, \infty}(M)$ ($\Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{m-q, \infty}(M)$) and d maps $\Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{q, \infty}(M)$ ($\Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{q, \infty}(M)$) into $\Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{q+1, \infty}(M)$ ($\Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{q+1, \infty}(M)$).

Definition 2.3. We define projections $\tilde{\mathcal{P}}_0, \tilde{\mathcal{P}}_1 : \Omega^\bullet(Y) \oplus \Omega^\bullet(Y) \rightarrow \Omega^\bullet(Y) \oplus \Omega^\bullet(Y)$ as follows. For $\phi \in \Omega^q(M, E)$

$$\tilde{\mathcal{P}}_0(\phi|_Y) = \begin{cases} \mathcal{P}_{-, \mathcal{L}_0}(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{+, \mathcal{L}_1}(\phi|_Y) & \text{if } q \text{ is odd,} \end{cases} \quad \tilde{\mathcal{P}}_1(\phi|_Y) = \begin{cases} \mathcal{P}_{+, \mathcal{L}_1}(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{-, \mathcal{L}_0}(\phi|_Y) & \text{if } q \text{ is odd.} \end{cases}$$

Then the above argument leads to the following cochain complexes

$$(\Omega_{\tilde{\mathcal{P}}_0}^{\bullet, \infty}(M), d) : 0 \longrightarrow \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{0, \infty}(M) \xrightarrow{d} \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{1, \infty}(M) \xrightarrow{d} \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{2, \infty}(M) \xrightarrow{d} \dots \longrightarrow 0. \quad (2.7)$$

$$(\Omega_{\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M), d) : 0 \longrightarrow \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{0, \infty}(M) \xrightarrow{d} \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{1, \infty}(M) \xrightarrow{d} \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{2, \infty}(M) \xrightarrow{d} \dots \longrightarrow 0. \quad (2.8)$$

We define the Laplacians $\Delta_{\tilde{\mathcal{P}}_0}^q$ and $\Delta_{\tilde{\mathcal{P}}_1}^q$ by

$$\Delta^q := d_q^* d_q + d_{q-1} d_{q-1}^*, \quad \text{Dom}(\Delta_{\tilde{\mathcal{P}}_0}^q) = \Omega_{\tilde{\mathcal{P}}_0}^{q, \infty}(M) = \begin{cases} \Omega_{\mathcal{P}_{-, \mathcal{L}_0}}^{q, \infty}(M) & \text{for } q \text{ even} \\ \Omega_{\mathcal{P}_{+, \mathcal{L}_1}}^{q, \infty}(M) & \text{for } q \text{ odd.} \end{cases}$$

We define $\text{Dom}(\Delta_{\tilde{\mathcal{P}}_1}^q)$ in the same way. It is not difficult to see that $\mathcal{P}_{-, \mathcal{L}_0}$ and $\mathcal{P}_{+, \mathcal{L}_1}$ are well-posed boundary conditions for the odd signature operator and Laplacian in the sense of Seeley ([7], [11]). We refer to Lemma 2.15 in [8] for details. Hence, $\Delta_{\tilde{\mathcal{P}}_0}^q$ and $\Delta_{\tilde{\mathcal{P}}_1}^q$ have compact resolvents and discrete spectra. Moreover, the Green formula shows that $\Delta_{\tilde{\mathcal{P}}_0}^q$ and $\Delta_{\tilde{\mathcal{P}}_1}^q$ are formally self-adjoint and non-negative. The following lemma is straightforward (see Lemma 2.11 in [8] for details).

Lemma 2.4. The cohomologies of the complex $(\Omega_{\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M), d)$ are given as follows.

$$\begin{aligned}H^q((\Omega_{\tilde{\mathcal{P}}_0}^{\bullet, \infty}(M), d)) &= \ker \Delta_{\tilde{\mathcal{P}}_0}^q = \begin{cases} H^q(M, Y) & \text{if } q \text{ is even} \\ H^q(M) & \text{if } q \text{ is odd,} \end{cases} \\ H^q((\Omega_{\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M), d)) &= \ker \Delta_{\tilde{\mathcal{P}}_1}^q = \begin{cases} H^q(M) & \text{if } q \text{ is even} \\ H^q(M, Y) & \text{if } q \text{ is odd.} \end{cases}\end{aligned} \quad (2.9)$$

Proof. We denote by $\mathcal{H}_{\text{rel}}^q(M) := \{\phi = \phi_1 + du \wedge \phi_2 \in \Omega^q(M) \mid d\phi = d^*\phi = 0, \phi_1|_Y = 0\}$ the space of harmonic q -forms satisfying the relative boundary condition. It is well known that $\mathcal{H}_{\text{rel}}^q(M)$ is isomorphic to the singular cohomology $H^q(M, Y)$. The Green theorem shows that $\ker \Delta_{\mathcal{P}_{-, \mathcal{L}_0}}^q = \{\phi \in \Omega^q(M) \mid d\phi = d^*\phi = 0, \mathcal{P}_{-, \mathcal{L}_0}(\phi|_Y) = 0\}$. We are going to show that $\ker \Delta_{\mathcal{P}_{-, \mathcal{L}_0}}^q = \mathcal{H}_{\text{rel}}^q(M)$. Let $\phi = \phi_1 + du \wedge \phi_2 \in \mathcal{H}_{\text{rel}}^q(M)$. Then by (2.1) with the fact that $\phi_1|_Y = 0$, we have $\phi|_Y = du \wedge (d^Y \psi_1 + \psi_2)$, which shows that $\mathcal{P}_{-, \mathcal{L}_0}(\phi|_Y) = 0$. Hence, $\phi \in \ker \Delta_{\mathcal{P}_{-, \mathcal{L}_0}}^q$. Conversely, let $\phi = \phi_1 + du \wedge \phi_2 \in \ker \Delta_{\mathcal{P}_{-, \mathcal{L}_0}}^q$. By (2.1) with the fact that $\mathcal{P}_{-, \mathcal{L}_0}(\phi|_Y) = 0$, we have $\phi|_Y = du \wedge (d^Y \psi_1 + \psi_2)$, which shows that $\phi \in \mathcal{H}_{\text{rel}}^q(M)$. Other cases can be checked in the same way. This completes the proof of the lemma. \square

In the next section, we discuss the Lefschetz fixed point formula on the complexes (2.7) and (2.8).

3. LEFSCHETZ FIXED POINT FORMULA ON THE COMPLEX $(\Omega_{\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M), d)$

We recall that g^M is assumed to be a product metric near Y and begin with the following definition.

Definition 3.1. For a smooth map $f : M \rightarrow M$, f is said to satisfy the Condition A if on some collar neighborhood $[0, \epsilon) \times Y$ of Y , $f : [0, \epsilon) \times Y \rightarrow M$ is expressed by $f(u, y) = (cu, B(y))$, where c is a positive real number which is not equal to 1 and $B : (Y, g^Y) \rightarrow (Y, g^Y)$ is an isometry.

Remark: If $f : M \rightarrow M$ satisfies the Condition A, then all the fixed points in Y are attracting if $0 < c < 1$ and repelling if $c > 1$.

If f satisfies the Condition A, for $\omega = \omega_1 + du \wedge \omega_2$ on a collar neighborhood of Y , $f^*\omega = B^*\omega_1 + cdu \wedge B^*\omega_2$. Since B is an isometry, B^* maps $\text{Im } d^Y$ and $\text{Im}(d^Y)^*$ onto $\text{Im } d^Y$ and $\text{Im}(d^Y)^*$, respectively. The following lemma shows that f^* maps $\Omega_{\tilde{\mathcal{P}}_0}^{\bullet, \infty}(M)$ into $\Omega_{\tilde{\mathcal{P}}_0}^{\bullet, \infty}(M)$ and maps $\Omega_{\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M)$ into $\Omega_{\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M)$.

Lemma 3.2. B^* maps \mathcal{K}^q onto \mathcal{K}^q and $\star_Y \mathcal{K}^q$ onto $\star_Y \mathcal{K}^q$.

Proof. Since B is an isometry, it is enough to show that B^* maps \mathcal{K}^q into \mathcal{K}^q . The following commutative diagrams show that for $[\omega] \in H^q(M)$, $B^*\iota^*\omega = \iota^*f^*\omega$.

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & M \\ B \downarrow & \circlearrowleft & \downarrow f \\ Y & \xrightarrow{\iota} & M \end{array} \implies \begin{array}{ccc} H^q(M) & \xrightarrow{\iota^*} & H^q(Y) \\ f^* \downarrow & \circlearrowleft & \downarrow B^* \\ H^q(M) & \xrightarrow{\iota^*} & H^q(Y) \end{array}$$

This fact together with Lemma 2.2 implies the result. \square

Since f^* commutes with d , $f^* : (\Omega_{\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M), d) \rightarrow (\Omega_{\tilde{\mathcal{P}}_0/\tilde{\mathcal{P}}_1}^{\bullet, \infty}(M), d)$ is a cochain map. In this section we are going to discuss the Lefschetz fixed point formula on these complexes for smooth maps having only simple fixed points and satisfying the Condition A.

Definition 3.3. Suppose that $f : M \rightarrow M$ is a smooth map satisfying the Condition A. We define the Lefschetz number of f with respect to the complex $(\Omega_{\tilde{\mathcal{P}}_i}^{\bullet, \infty}(M), d)$ ($i = 0, 1$) by

$$L_{\tilde{\mathcal{P}}_i}(f) = \sum_{q=0}^m (-1)^q \text{Tr} \left(f^* : H^q((\Omega_{\tilde{\mathcal{P}}_i}^{\bullet, \infty}(M), d)) \rightarrow H^q((\Omega_{\tilde{\mathcal{P}}_i}^{\bullet, \infty}(M), d)) \right).$$

We are going to express $L_{\widehat{\mathcal{P}}_i}(f)$ in terms of fixed points of f and some additional data. We consider $L_{\widehat{\mathcal{P}}_0}(f)$ first. Using Lemma 2.4 and the standard argument for the trace of a heat operator (see Lemma 1.10.1 in [6] or Theorem 4 in [3] for details), we have

$$\begin{aligned}
L_{\widehat{\mathcal{P}}_0}(f) &= \sum_{q=\text{even}} \text{Tr}(f^* : H^q(M, Y) \rightarrow H^q(M, Y)) - \sum_{q=\text{odd}} \text{Tr}(f^* : H^q(M) \rightarrow H^q(M)) \\
&= \sum_{q=0}^m (-1)^q \text{Tr}(f^* e^{-t\Delta_{\widehat{\mathcal{P}}_0}^q}) = \lim_{t \rightarrow 0} \sum_{q=0}^m (-1)^q \text{Tr}(f^* e^{-t\Delta_{\widehat{\mathcal{P}}_0}^q}) \\
&= \lim_{t \rightarrow 0} \left\{ \sum_{q=\text{even}} \text{Tr}(f^* e^{-t\Delta_{\mathcal{P}_-, \mathcal{L}_0}^q}) - \sum_{q=\text{odd}} \text{Tr}(f^* e^{-t\Delta_{\mathcal{P}_+, \mathcal{L}_1}^q}) \right\} \\
&= \lim_{t \rightarrow 0} \int_M \left\{ \sum_{q=\text{even}} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^q(t, f(x), x)) - \sum_{q=\text{odd}} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^q(t, f(x), x)) \right\} d\text{vol}(x),
\end{aligned} \tag{3.1}$$

where $\mathcal{T}_q(x) := \Lambda^q((df(x))^T) : \Lambda^q T_{f(x)}^* M \rightarrow \Lambda^q T_x^* M$ is the pull-back map mapping the fiber over $f(x)$ to the fiber over x and $\mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0 / \mathcal{P}_+, \mathcal{L}_1}^q(t, x, z)$ is the kernel of $e^{-t\Delta_{\mathcal{P}_-, \mathcal{L}_0 / \mathcal{P}_+, \mathcal{L}_1}^q}$. We choose $\epsilon > 0$ such that $([0, 2\epsilon] \times Y) \cap \mathcal{F}(f) = \mathcal{F}_Y(f)$. For each $x \in \mathcal{F}_0(f)$, choose a small open neighborhood U_x of x such that $U_x \cap ([0, \epsilon] \times Y) = \emptyset$. Putting $W := M - (\cup_{x \in \mathcal{F}_0(f)} U_x \cup [0, \frac{\epsilon}{7}] \times Y)$, the standard argument (see Lemma 1.10.2 in [6] or Theorem 5 in [3] for details) shows that

$$\lim_{t \rightarrow 0} \int_W \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0 / \mathcal{P}_+, \mathcal{L}_1}^q(t, f(x), x)) d\text{vol}(x) = 0. \tag{3.2}$$

Hence, we can rewrite (3.1) as follows.

$$\begin{aligned}
L_{\widehat{\mathcal{P}}_0}(f) &= \lim_{t \rightarrow 0} \sum_{x \in \mathcal{F}_0(f)} \sum_{q=\text{even}} \int_{U_x} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^q(t, f(x), x)) d\text{vol}(x) \\
&\quad - \lim_{t \rightarrow 0} \sum_{x \in \mathcal{F}_0(f)} \sum_{q=\text{odd}} \int_{U_x} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^q(t, f(x), x)) d\text{vol}(x) \\
&\quad + \lim_{t \rightarrow 0} \sum_{q=\text{even}} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^q(t, f(x), x)) du d\text{vol}(y) \\
&\quad - \lim_{t \rightarrow 0} \sum_{q=\text{odd}} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^q(t, f(x), x)) du d\text{vol}(y).
\end{aligned} \tag{3.3}$$

We next construct the parametrix $Q_{\mathcal{P}_-, \mathcal{L}_0 / \mathcal{P}_+, \mathcal{L}_1}^q(t, x, z)$ of the heat kernel $\mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0 / \mathcal{P}_+, \mathcal{L}_1}^q(t, x, z)$ by combining the interior contribution and the boundary contribution. We denote by \widetilde{M} the closed double of M , i.e., $\widetilde{M} = M \cup_Y M$ and extend the Laplacian Δ^q on M to the Laplacian on \widetilde{M} , denoted by $\widetilde{\Delta}^q$. Let $\widetilde{\mathcal{E}}_q(t, x, z)$ be the kernel of the heat operator $e^{-t\widetilde{\Delta}^q}$. It is well known (for example, p.225 in [4]) that

$$|\widetilde{\mathcal{E}}_q(t, x, z)| \leq c_1 t^{-\frac{m}{2}} e^{-c_2 \frac{d(x, z)^2}{t}}, \tag{3.4}$$

where c_i 's are some positive constants.

Let $N_\infty := [0, \infty) \times Y$ be a half infinite cylinder and $\Delta_{N_\infty}^q := -\partial_u^2 + \begin{pmatrix} \Delta_Y^q \\ \Delta_Y^{q-1} \end{pmatrix}$ be the Laplacian acting on q -forms on N_∞ . We decompose $\Omega^q(Y)$ by $\Omega^q(Y) = \Omega_-^q(Y) \oplus \Omega_+^q(Y)$, where

$$\Omega_-^q(Y) := (\text{Im } d^Y \oplus \mathcal{K}) \cap \Omega^q(Y), \quad \Omega_+^q(Y) := (\text{Im}(d^Y)^* \oplus \star_Y \mathcal{K}) \cap \Omega^q(Y). \quad (3.5)$$

We denote by $\{\phi_{q,j} \mid j = 1, 2, \dots\}$ and $\{\psi_{q,j} \mid j = 1, 2, \dots\}$ the orthonormal bases of $\Omega_-^q(Y)$ and $\Omega_+^q(Y)$ consisting of eigenforms of Δ_Y^q with eigenvalues $\{\lambda_{q,j} \mid j = 1, 2, \dots\}$ and $\{\mu_{q,j} \mid j = 1, 2, \dots\}$, respectively. Then the heat kernels $\mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^{\text{cyl}, q}$ and $\mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^{\text{cyl}, q}$ of $\Delta_{N_\infty}^q$ with respect to the boundary conditions $\mathcal{P}_-, \mathcal{L}_0$ and $\mathcal{P}_+, \mathcal{L}_1$ on $\{0\} \times Y$ are given as follows (cf. p.226 in [4]).

$$\begin{aligned} \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^{\text{cyl}, q}(t, (u, y), (v, y')) &= \sum_{j=1}^{\infty} \frac{e^{-t\lambda_{q,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right) \phi_{q,j}(y) \otimes \phi_{q,j}^*(y') \\ &+ \sum_{j=1}^{\infty} \frac{e^{-t\mu_{q,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right) \psi_{q,j}(y) \otimes \psi_{q,j}^*(y') \\ &+ \sum_{j=1}^{\infty} \frac{e^{-t\lambda_{q-1,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right) (du \wedge \phi_{q-1,j}(y)) \otimes (dv \wedge \phi_{q-1,j}(y'))^* \\ &+ \sum_{j=1}^{\infty} \frac{e^{-t\mu_{q-1,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right) (du \wedge \psi_{q-1,j}(y)) \otimes (dv \wedge \psi_{q-1,j}(y'))^*, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^{\text{cyl}, q}(t, (u, y), (v, y')) &= \sum_{j=1}^{\infty} \frac{e^{-t\lambda_{q,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right) \phi_{q,j}(y) \otimes \phi_{q,j}^*(y') \\ &+ \sum_{j=1}^{\infty} \frac{e^{-t\mu_{q,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right) \psi_{q,j}(y) \otimes \psi_{q,j}^*(y') \\ &+ \sum_{j=1}^{\infty} \frac{e^{-t\lambda_{q-1,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right) (du \wedge \phi_{q-1,j}(y)) \otimes (dv \wedge \phi_{q-1,j}(y'))^* \\ &+ \sum_{j=1}^{\infty} \frac{e^{-t\mu_{q-1,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right) (du \wedge \psi_{q-1,j}(y)) \otimes (dv \wedge \psi_{q-1,j}(y'))^*. \end{aligned} \quad (3.7)$$

Let $\rho(a, b)$ be a smooth increasing function of real variable such that

$$\rho(a, b)(u) = \begin{cases} 0 & \text{for } u \leq a \\ 1 & \text{for } u \geq b \end{cases}.$$

We put

$$\phi_1 := 1 - \rho\left(\frac{5\epsilon}{7}, \frac{6\epsilon}{7}\right), \quad \psi_1 := 1 - \rho\left(\frac{3\epsilon}{7}, \frac{4\epsilon}{7}\right), \quad \phi_2 := \rho\left(\frac{\epsilon}{7}, \frac{2\epsilon}{7}\right), \quad \psi_2 := \rho\left(\frac{3\epsilon}{7}, \frac{4\epsilon}{7}\right),$$

and

$$\begin{aligned} \mathcal{Q}_{\mathcal{P}_-, \mathcal{L}_0}^q(t, (u, y), (v, y')) &= \phi_1(u) \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^{\text{cyl}, q}(t, (u, y), (v, y')) \psi_1(v) + \phi_2(u) \tilde{\mathcal{E}}^q(t, (u, y), (v, y')) \psi_2(v), \\ \mathcal{Q}_{\mathcal{P}_+, \mathcal{L}_1}^q(t, (u, y), (v, y')) &= \phi_1(u) \mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^{\text{cyl}, q}(t, (u, y), (v, y')) \psi_1(v) + \phi_2(u) \tilde{\mathcal{E}}^q(t, (u, y), (v, y')) \psi_2(v). \end{aligned} \quad (3.8)$$

Then, $\mathcal{Q}_{\mathcal{P}_{-, \mathcal{L}_0}}^q$ and $\mathcal{Q}_{\mathcal{P}_{+, \mathcal{L}_1}}^q$ are parametrices for the kernels of $e^{-t\Delta_{\mathcal{P}_{-, \mathcal{L}_0}}^q}$ and $e^{-t\Delta_{\mathcal{P}_{+, \mathcal{L}_1}}^q}$, respectively. The standard computation using (3.4), (3.6) and (3.7) (see [2], [4] for details) shows that for $0 < t \leq 1$ and $\alpha = \mathcal{P}_{-, \mathcal{L}_0}$ or $\mathcal{P}_{+, \mathcal{L}_1}$, there exist some positive constants c_1 and c_2 such that

$$|\mathcal{E}_\alpha^q(t, (u, y), (v, y')) - \mathcal{Q}_\alpha^q(t, (u, y), (v, y'))| \leq c_1 e^{-\frac{c_2}{t}}, \quad (3.9)$$

which shows that

$$\lim_{t \rightarrow 0} (\mathcal{E}_\alpha^q(t, (u, y), (v, y')) - \mathcal{Q}_\alpha^q(t, (u, y), (v, y'))) = 0. \quad (3.10)$$

Hence, in view of (3.3) with $x \in \mathcal{F}_0(f)$, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{U_x} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_\alpha^q(t, f(x), x)) d\text{vol}(x) &= \lim_{t \rightarrow 0} \int_{U_x} \text{Tr}(\mathcal{T}_q(x) \mathcal{Q}_\alpha^q(t, f(x), x)) d\text{vol}(x) \\ &= \lim_{t \rightarrow 0} \int_{U_x} \text{Tr}(\mathcal{T}_q(x) \tilde{\mathcal{E}}^q(t, f(x), x)) d\text{vol}(x), \end{aligned} \quad (3.11)$$

which yields the following equalities.

$$\begin{aligned} &\lim_{t \rightarrow 0} \sum_{x \in \mathcal{F}_0(f)} \sum_{q=\text{even}} \int_{U_x} \text{Tr} \mathcal{T}_q(x) \left(\mathcal{E}_{\mathcal{P}_{-, \mathcal{L}_0}}^q(t, f(x), x) \right) d\text{vol}(x) \\ &- \lim_{t \rightarrow 0} \sum_{x \in \mathcal{F}_0(f)} \sum_{q=\text{odd}} \int_{U_x} \text{Tr} \mathcal{T}_q(x) \left(\mathcal{E}_{\mathcal{P}_{+, \mathcal{L}_1}}^q(t, f(x), x) \right) d\text{vol}(x) \\ &= \lim_{t \rightarrow 0} \sum_{x \in \mathcal{F}_0(f)} \sum_{q=0}^m (-1)^q \int_{U_x} \text{Tr} \left(\mathcal{T}_q(x) \tilde{\mathcal{E}}^q(t, f(x), x) \right) d\text{vol}(x) = \sum_{x \in \mathcal{F}_0(f)} \text{sign det}(I - df(x)), \end{aligned} \quad (3.12)$$

where we refer to Theorem 1.10.4 in [6] or Theorem 10.12 in [10] for the proof of the last equality.

We next analyze the boundary contribution. For $\alpha = \mathcal{P}_{-, \mathcal{L}_0}$ or $\mathcal{P}_{+, \mathcal{L}_1}$, by (3.10) we have

$$\begin{aligned} &\lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_\alpha^q(t, f(x), x)) du d\text{vol}(y) = \lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr}(\mathcal{T}_q(x) \mathcal{Q}_\alpha^q(t, f(x), x)) du d\text{vol}(y) \\ &= \lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr}(\mathcal{T}_q(x) \mathcal{E}_\alpha^{\text{cyl}, q}(t, f(x), x)) du d\text{vol}(y). \end{aligned} \quad (3.13)$$

We note that on $[0, \frac{\epsilon}{7}] \times Y$, f is assumed to be $f(u, y) = (c u, B(y))$, where $B : (Y, g^Y) \rightarrow (Y, g^Y)$ is an isometry. Let us consider the case of $\alpha = \mathcal{P}_{-, \mathcal{L}_0}$. We can treat the case of $\alpha = \mathcal{P}_{+, \mathcal{L}_1}$ in the same way. Put $x = (u, y)$ and $\mathfrak{B}_q(y) := \Lambda^q((d^Y B(y))^T)$. Since $\mathcal{T}_q(u, y) \phi_{q,j}(B(y)) = \mathfrak{B}_q(y) \phi_{q,j}(B(y))$, we have

$$\begin{aligned} &\lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \sum_{j=1}^{\infty} \frac{e^{-t\lambda_{q,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(c-1)^2 u^2}{4t}} - e^{-\frac{(c+1)^2 u^2}{4t}} \right) \langle \mathfrak{B}_q(y) \phi_{q,j}(B(y)), \phi_{q,j}(y) \rangle du d\text{vol}(y) \\ &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{\pi}} \int_0^{\frac{\epsilon}{14\sqrt{t}}} \left(e^{-(c-1)^2 x^2} - e^{-(c+1)^2 x^2} \right) dx \cdot \lim_{t \rightarrow 0} \int_Y \sum_{j=1}^{\infty} e^{-t\lambda_{q,j}} \langle \mathfrak{B}_q(y) \phi_{q,j}(B(y)), \phi_{q,j}(y) \rangle d\text{vol}(y) \\ &= \frac{1}{2} \left(\frac{1}{|1-c|} - \frac{1}{1+c} \right) \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^q} |_{\Omega_-^q(Y)} \right), \end{aligned} \quad (3.14)$$

where $\langle \cdot, \cdot \rangle$ is the pointwise inner product of differential forms induced by the metric g_Y . Similarly, since $\mathcal{T}_q(u, y)(du \wedge \phi_{q-1,j}(B(y))) = c du \wedge (\mathfrak{B}(y)\phi_{q-1,j}(B(y)))$, we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \sum_{j=1}^{\infty} \frac{e^{-t\lambda_{q,j}}}{\sqrt{4\pi t}} \left(e^{-\frac{(c-1)^2 u^2}{4t}} - e^{-\frac{(c+1)^2 u^2}{4t}} \right) \times \\ & \quad \langle cdu \wedge \mathfrak{B}_{q-1}(y)\phi_{q-1,j}(B(y)), du \wedge \phi_{q-1,j}(y) \rangle du dvol(y) \\ &= \frac{1}{2} \left(\frac{c}{|1-c|} - \frac{c}{1+c} \right) \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_-^{q-1}(Y)} \right). \end{aligned} \quad (3.15)$$

Same computation using (3.6) shows that

$$\begin{aligned} & \lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr} \left(\mathcal{T}_q(u, y) \mathcal{E}_{\mathcal{P}_-, \mathcal{L}_0}^{\text{cyl}, q}(t, f(u, y), (u, y)) \right) du dvol(y) \\ &= \frac{1}{2} \left(\frac{1}{|1-c|} - \frac{1}{1+c} \right) \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^q}|_{\Omega_-^q(Y)} \right) + \frac{1}{2} \left(\frac{1}{|1-c|} + \frac{1}{1+c} \right) \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^q}|_{\Omega_+^q(Y)} \right) \\ &+ \frac{1}{2} \left(\frac{c}{|1-c|} - \frac{c}{1+c} \right) \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_-^{q-1}(Y)} \right) \\ &+ \frac{1}{2} \left(\frac{c}{|1-c|} + \frac{c}{1+c} \right) \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_+^{q-1}(Y)} \right) \\ &= \frac{1}{2|1-c|} \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^q} \right) + \frac{c}{2|1-c|} \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}} \right) \\ &\quad + \frac{1}{2(1+c)} \cdot \lim_{t \rightarrow 0} \left(\text{Tr} \left(B^* e^{-t\Delta_Y^q}|_{\Omega_+^q(Y)} \right) - \text{Tr} \left(B^* e^{-t\Delta_Y^q}|_{\Omega_-^q(Y)} \right) \right) \\ &\quad + \frac{c}{2(1+c)} \cdot \lim_{t \rightarrow 0} \left(\text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_+^{q-1}(Y)} \right) - \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_-^{q-1}(Y)} \right) \right). \end{aligned} \quad (3.16)$$

Similarly, using (3.7), we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr} \left(\mathcal{T}_q(u, y) \mathcal{E}_{\mathcal{P}_+, \mathcal{L}_1}^{\text{cyl}, q}(t, f(u, y), (u, y)) \right) du dvol(y) \\ &= \frac{1}{2|1-c|} \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^q} \right) + \frac{c}{2|1-c|} \cdot \lim_{t \rightarrow 0} \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}} \right) \\ &\quad - \frac{1}{2(1+c)} \cdot \lim_{t \rightarrow 0} \left(\text{Tr} \left(B^* e^{-t\Delta_Y^q}|_{\Omega_+^q(Y)} \right) - \text{Tr} \left(B^* e^{-t\Delta_Y^q}|_{\Omega_-^q(Y)} \right) \right) \\ &\quad - \frac{c}{2(1+c)} \cdot \lim_{t \rightarrow 0} \left(\text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_+^{q-1}(Y)} \right) - \text{Tr} \left(B^* e^{-t\Delta_Y^{q-1}}|_{\Omega_-^{q-1}(Y)} \right) \right). \end{aligned} \quad (3.17)$$

Finally, combining (3.16) and (3.17), we have

$$\begin{aligned}
& \lim_{t \rightarrow 0} \sum_{q=\text{even}} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr} \left(\mathcal{T}_q(u, y) \mathcal{E}_{\mathcal{P}^-, \mathcal{L}_0}^{\text{cyl}, q}(t, f(u, y), (u, y)) \right) du \, d\text{vol}(y) \\
& - \lim_{t \rightarrow 0} \sum_{q=\text{odd}} \int_Y \int_0^{\frac{\epsilon}{7}} \text{Tr} \left(\mathcal{T}_q(u, y) \mathcal{E}_{\mathcal{P}^+, \mathcal{L}_1}^{\text{cyl}, q}(t, f(u, y), (u, y)) \right) du \, d\text{vol}(y) \\
& = \frac{1-c}{2|1-c|} \cdot \lim_{t \rightarrow 0} \sum_{q=0}^{m-1} (-1)^q \text{Tr} \left(B^* e^{-t\Delta_Y^q} \right) \\
& + \lim_{t \rightarrow 0} \frac{1}{2} \sum_{q=0}^{m-1} \left(\text{Tr} \left(B^* e^{-t\Delta_Y^q} |_{\Omega_+^q(Y)} \right) - \text{Tr} \left(B^* e^{-t\Delta_Y^q} |_{\Omega_-^q(Y)} \right) \right). \tag{3.18}
\end{aligned}$$

Using (3.5) and the following commutative diagram

$$\begin{array}{ccc}
\text{Im}(d^Y)^* \cap \Omega^q(Y) & \xrightarrow{d^Y} & \text{Im } d^Y \cap \Omega^{q+1}(Y) \\
B^* e^{-t\Delta_Y} \downarrow & \circlearrowleft & \downarrow B^* e^{-t\Delta_Y} \\
\text{Im}(d^Y)^* \cap \Omega^q(Y) & \xrightarrow{d^Y} & \text{Im } d^Y \cap \Omega^{q+1}(Y)
\end{array}$$

with the fact that $\text{sign det}(I - df(y)) = \text{sign}(1 - c) \cdot \text{sign det}(I - df_Y(y))$, we can rewrite (3.18) by

$$(3.18) = \frac{1}{2} \sum_{y \in \mathcal{F}_Y(f)} \text{sign det}(I - df(y)) + \frac{1}{2} \{ \text{Tr}(B^* : (\star_Y \mathcal{K}) \rightarrow (\star_Y \mathcal{K})) - \text{Tr}(B^* : \mathcal{K} \rightarrow \mathcal{K}) \}.$$

Furthermore, $\frac{1}{2} \{ \text{Tr}(B^* : (\star_Y \mathcal{K}) \rightarrow (\star_Y \mathcal{K})) - \text{Tr}(B^* : \mathcal{K} \rightarrow \mathcal{K}) \}$ is equal to 0 if $B : (Y, g^Y) \rightarrow (Y, g^Y)$ is orientation preserving and is equal to $-\text{Tr}(B^* : \mathcal{K} \rightarrow \mathcal{K})$ if B is orientation reversing. We can compute $L_{\widehat{\mathcal{P}}_1}(f)$ in the same way. Summarizing the above arguments with Lemma 2.2, we have the following result, which is the main result of this paper.

Theorem 3.4. *Let (M, Y, g^M) be an m -dimensional compact oriented Riemannian manifold with boundary Y and g^M be assumed to be a product metric near Y . Suppose that $f : M \rightarrow M$ is a smooth map having only simple fixed points and satisfying the condition A. Then the following equalities hold.*

$$\begin{aligned}
(1) \quad & \sum_{q=\text{even}} \text{Tr}(f^* : H^q(M, Y) \rightarrow H^q(M, Y)) - \sum_{q=\text{odd}} \text{Tr}(f^* : H^q(M) \rightarrow H^q(M)) \\
& = \sum_{x \in \mathcal{F}_0(f)} \text{sign det}(I - df(x)) + \frac{1}{2} \sum_{y \in \mathcal{F}_Y(f)} \text{sign det}(I - df(y)) - K_0 \\
(2) \quad & \sum_{q=\text{even}} \text{Tr}(f^* : H^q(M) \rightarrow H^q(M)) - \sum_{q=\text{odd}} \text{Tr}(f^* : H^q(M, Y) \rightarrow H^q(M, Y)) \\
& = \sum_{x \in \mathcal{F}_0(f)} \text{sign det}(I - df(x)) + \frac{1}{2} \sum_{y \in \mathcal{F}_Y(f)} \text{sign det}(I - df(y)) + K_0,
\end{aligned}$$

where $K_0 = 0$ if B is orientation preserving and $K_0 = \text{Tr}(B^* : \text{Im } \iota^* \rightarrow \text{Im } \iota^*)$ with $\iota^* : H^\bullet(M) \rightarrow H^\bullet(Y)$ if B is orientation reversing.

Combining this result with (1.2), we have the following result.

Corollary 3.5. *We assume the same assumptions as in Theorem 3.4. Then :*

$$\begin{aligned}
(1) \quad & \sum_{q=\text{even}} \text{Tr}(f^* : H^q(M) \rightarrow H^q(M)) - \sum_{q=\text{even}} \text{Tr}(f^* : H^q(M, Y) \rightarrow H^q(M, Y)) \\
&= \frac{1}{2} \sum_{y \in \mathcal{F}_Y^+(f)} \text{sign det}(I - df(y)) - \frac{1}{2} \sum_{y \in \mathcal{F}_Y^-(f)} \text{sign det}(I - df(y)) + K_0, \\
(2) \quad & \sum_{q=\text{odd}} \text{Tr}(f^* : H^q(M) \rightarrow H^q(M)) - \sum_{q=\text{odd}} \text{Tr}(f^* : H^q(M, Y) \rightarrow H^q(M, Y)) \\
&= -\frac{1}{2} \sum_{y \in \mathcal{F}_Y^+(f)} \text{sign det}(I - df(y)) + \frac{1}{2} \sum_{y \in \mathcal{F}_Y^-(f)} \text{sign det}(I - df(y)) + K_0,
\end{aligned}$$

where either $\mathcal{F}_Y^+(f) = \emptyset$ or $\mathcal{F}_Y^-(f) = \emptyset$, depending on c in the Condition A.

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